Rectangular billiard in the presence of a flux line

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We analyze the quantum spectrum of a rectangular billiard coupled to flux line. The nearest neighbor spacing distribution shows a transition from Poisson to almost Wigner distribution as the flux parameter is varied even though classically the system remains pseudointegrable and nonchaotic. In fact, the level statistics do not display any generic behavior. The calculations presented here are simple and avoid the complexity involved in modeling the singular quantum billiard problem.

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I. INTRODUCTION

The statistical properties of quantum spectra of classically chaotic systems has attracted much interest in recent years. One method of analyzing the quantum spectrum is through the analysis of nearest neighbor spacing (NNS) distributions. For a given sequence of levels $\{E_i\}$, the NNS is defined through the unfolded level spacings s_i,

$$s_i \approx (E_{i+1} - E_i)g(E_i), \tag{1}$$

where $g(E_i)$ is the average density of the quantum states evaluated at energy E_i. The spacings s_i, thus generated, are distributed in various bins to yield the so-called $P(s)$ distribution. It is conjectured that $P(s)$ distribution is Poisson for classically integrable systems and Wigner (Gaussian orthogonal ensemble, GOE) for time-reversal (T) invariant classically chaotic systems [1] and has been confirmed by various numerical studies [2]. However, exceptions to this conjecture have been pointed out by several authors, the most dramatic one being the occurrence of Wigner distribution for a one-dimensional system, that is obviously integrable [3].

There are, however, classical systems that are not fully integrable nor do they display a fully developed classical chaos, so-called pseudointegrable systems [4] for which some exact results have been derived only recently [5]. The level statistics of these systems, in general, do not conform to a generic situation. In particular there is a growing interest on the singular billiard [6–8] where the spectral statistics obeys Wigner distribution despite the fact that the underlying dynamics is regular except at one point (measure zero). It has been, therefore, remarked in this context that the classical dynamics has no correlation with the level statistics of the quantum problem [7], since the semiclassical limit is valid only as $E \to \infty$ [6]. Obtaining the spectrum of the Hamiltonian of a singular billiard, however, is a difficult task and one has to employ the self-adjoint extension theory [6]. In addition, the origin and the physical interpretation of the coupling parameter that arises in this context remains obscure. In what follows we explore the relation between classical integrability properties and the quantum NNS distribution in the context of the so-called Aharanov-Bohm billiard where the computation can be done without the complicated machinery needed for the singular billiard.

Our system consists of a particle in a rectangular enclosure in two dimensions and coupled to a flux line perpendicular to the plane of motion located at the origin which is taken to be the center of the rectangle. This is, of course, equivalent to a charged particle enclosed by a rectangular boundary interacting with magnetic field of an infinitely thin and long solenoid enclosing a finite flux. There are several points in favor of considering this system. (i) The system belongs to the class of the so-called pseudointegrable systems, i.e., systems that are classically integrable but are *not* integrable by action-angle coordinates [4,9]. The system is not chaotic and while it shares common features with the singular billiards it is much simpler to analyze both analytically and numerically. (ii) The symmetries of the system are also interesting in that while the time-reversal invariance is not respected a combination of time reversal and space reflection is respected. Consequently, it does admit an antiunitary symmetry. The relevence of such a symmetry has been noted by Robnik and Berry [10] when the system is classically chaotic in addition. (iii) In this system the pseudointegrability aspects and the boundary aspects are well seperated. Consequently, it is straightforward to consider variations of the boundary and/or simple modifications of the Hamiltonian. This allows one to explore the relation of pseudointegrability and the NNS distribution, if there is any, more readily. (iv) While the system has been considered previously in a variety of theoretical contexts [10–12], recent experimental advances with quantum dots may, in fact, make it experimentally accessible.

However, our principle reason to study this system is because it is simple and flexible enough to allow the exploration of any possible connection between classical properties and their quantum manifestation as reflected in the NNS distribution.

The paper is organized as follows: In Sec. II we discuss the classical aspects of the system including in particular the effect of the flux line on classical orbits both in the configuration space and phase space. We also explain the notion of pseudointegrability. In Sec. III we discuss the symmetries and use them to simplify the diagonalization problem. We describe our computations and present our results on the quantum spectrum and the NNS distribution. The last section contains a discussion of our results together with a comparison with the Aharanov-Bohm billiard in other contexts.

II. CLASSICAL ANALYSIS

The two-dimensional Lagrangian describing the motion of a particle coupled to a flux line at the origin is given by,

$$L(x,y,\dot{x},\dot{y}) = \frac{1}{2}[\dot{x}^2+\dot{y}^2] + \alpha\frac{x\dot{y}-y\dot{x}}{x^2+y^2}, \tag{2}$$

where α (which is dimensionless) represents the coupling of the particle to the flux line. We have set the mass of the particle $m = 1$ (and $\hbar = 1 = c$ hereafter). In polar coordinates,

$$L = \frac{1}{2}[\dot{r}^2 + r^2\dot{\theta}^2] + \alpha\dot{\theta}. \tag{3}$$

The classical action is manifestly invariant under the combined operation of time reversal and parity (reflection of the x or y axis). This is manifested as the antiunitary symmetry at the quantum level as alluded to in the introduction. Since L changes only by a total derivative, the Euler-Lagrange equations of motion are unchanged, that is, $\ddot{x} = 0 = \ddot{y}$. The canonical momentum $\vec{p}$ is given by,

$$p_x = \dot{x} - \alpha\frac{y}{r^2}, \tag{4}$$

$$p_y = \dot{y} + \alpha\frac{x}{r^2}, \tag{5}$$

and the Hamiltonian may be written as,

$$H = \frac{1}{2}[\vec{p}+\vec{A}]^2 = \frac{1}{2}(p_x^2+p_y^2) + \alpha\frac{yp_x - xp_y}{r^2} + \alpha^2\frac{1}{2r^2}, \tag{6}$$

where

$$A_i = \alpha\epsilon^{ij}r_j/r^2 \quad , \quad \epsilon_{12} = -\epsilon_{21} = 1$$

which is like a vector potential (Aharanov-Bohm-type). The corresponding magnetic field $B = \alpha\delta^2(\vec{r})$ is singular at the origin. The flux α is, however, finite but arbitrary and does not affect the quantization even when it is not an integer. Notice that from equations of motion, there are still two constants of motion, namely $\dot{x}^2$ and $\dot{y}^2$ (not $\dot{x}$ and $\dot{y}$ as the equations of motion suggest since at every reflection there is a change of sign). However the Poisson brackets $\{H,(p_x+A_x)^2\}$ and $\{H,(p_y+A_y)^2\}$ vanish everywhere except at the origin where they diverge. (It is in this sense that we call the system pseudointegrable.) In other words, if we took the usual phase space of a particle in a rectangle and remove the region $\vec{r} = \vec{0}$ (a two-dimensional plane in the phase space) then the dynamical system is integrable in that there exist two constants of motion in involution everywhere in the phase space.

The removal of regions has the following effect on the classical orbits in the configuration space. The Euler-Langrange equations being independent of α imply that classical trajectories are straight lines reflecting at the walls. The orbits that avoid the origin are completely unaffected by the α dependent term. However, those classical orbits that head towards the origin get reflected back. This can be seen explicitly by using a smeared δ function for the magnetic field. For example by taking

$$A_r = 0, \quad A_\theta = \frac{\alpha}{r}(1 - e^{-\frac{r^2}{\epsilon}}), \tag{7}$$

the magnetic field is given by,

$$B(r) = \frac{2\alpha}{\epsilon}e^{-\frac{r^2}{\epsilon}}.$$

This reduces to the case under consideration when ϵ is taken to zero. By considering orbits that begin for "large r" and head towards the origin and computing their scattering angle in the limit of $\epsilon \to 0$ one sees that the scattering angle is π. This is also independent of α as long as $\alpha \neq 0$.

Thus for nonzero α one can think of the system as a rectangular billiard with a reflecting point at the origin. This feature is independent of α. For all nonzero values of α precisely the same set of classical orbits is affected in precisely the same α independent manner. Note however that the classical action does have an explicit α dependence.

In the phase space the orbits themselves do depend on α. The Hamilton's equations of motion are

$$\dot{p}_x = 2\alpha\frac{xyp_x}{r^4} - 2\alpha\frac{x^2p_y}{r^4} + \alpha\frac{p_y}{r^2} + \alpha^2\frac{x}{r^4}, \tag{8}$$

$$\dot{p}_y = -2\alpha\frac{xyp_y}{r^4} + 2\alpha\frac{y^2p_x}{r^4} - \alpha\frac{p_x}{r^2} + \alpha^2\frac{y}{r^4}, \tag{9}$$

and $\dot{x}, \dot{y}$ are given by Eq. (4) and Eq. (5). From the equations of motion in phase space it follows,

$$x(t) = x_0 + u_x t, \tag{10}$$

$$y(t) = y_0 + u_y t, \tag{11}$$

$$p_x(t) = u_x - \alpha\frac{y(t)}{x^2(t)+y^2(t)}, \tag{12}$$

$$p_y(t) = u_y + \alpha\frac{x(t)}{x^2(t)+y^2(t)}. \tag{13}$$

Therefore, the phase space trajectories depend on α. Notice that the orbits that do not pass through $\vec{r} = \vec{0}$ and are periodic in configuration space are also periodic in phase space. However periodic orbits in configuration space which reflect at $\vec{r} = \vec{0}$ are not periodic orbits in phase space.

Thus even for classical phase space orbits we can see that for all $\alpha \neq 0$ exactly the same class of classical orbits are affected though they are now affected in an α dependent way. This indeed shows up in the quantum spectrum as we will see in the next section.

III. SPECTRAL STATISTICS

We now study the quantum spectrum of the system when the particle is confined within rectangular boundaries, $-a \leq x \leq a$ and $-b \leq y \leq b$. Similar quantal studies on the singular rectangular billiards have lead to anomalies in the spectral statistics as mentioned earlier. Unlike these systems, where one needs to model the singular interaction, we have a continuous parameter α through which we can analyze the spectral flow and statistics. First, a few observations.

Let $H(\alpha)$ be the Hamiltonian of Eq. (6) and let P denote either of the unitary operators implementing the reflections in the x and y axis. It follows immediately that,

$$P\ H(\alpha)\ P^{\dagger} = H(-\alpha). \tag{14}$$

Therefore, if ψ is an eigenfunction of $H(\alpha)$ with eigenvalue E, then $P(\psi)$ is an eigenfunction of $H(-\alpha)$ with the same eigenvalue and conversely. Since the Hamiltonian is a positive definite operator, there are no zero eigenvalues and the spectra of $H(\alpha)$ and $H(-\alpha)$ are identical. We can, therefore, confine our attention to positive values of α (say).

The Hamiltonian of the system [Eq. (6)] is not invariant under either time reversal or parity but is invariant under a combined operation of both. This has its implications on the spectrum of states. For any value of α, the Hamiltonian is rotationally invariant but the boundary is not. However, rotation through π or equivalently inversion leaves both the Hamiltonian and the boundary invariant. Consequently, the state space $\mathcal{H}$ can be expressed as

$$\mathcal{H} = \mathcal{H}_{\text{even}} \oplus \mathcal{H}_{\text{odd}}, \tag{15}$$

where $\mathcal{H}_{\text{even}}$ and $\mathcal{H}_{\text{odd}}$ are subspaces with "inversion parity" $+1$ and -1, respectively. The Hamiltonian for each α can, therefore, be diagonalized in each of the subspaces separately. One can easily construct unitary operators that will map the even and the odd subspaces onto each other. For instance, multiplication by $e^{ik\theta}$ will interchange the subspaces for any odd integer k while for even integer k it will leave each subspace invariant. Here θ is the polar coordinate.

Next we observe that

$$H(\alpha)\ [\ e^{i\beta\theta}\ \psi(x,y)\ ] = e^{i\beta\theta}\ [\ H(\alpha-\beta)\ \psi(x,y)\ ]. \tag{16}$$

Choosing β to be $\pm$ 1 or $\pm$ 2 one can see the following properties of the eigenvalues of the Hamiltonian:

$$E_n^{\text{even}}(\alpha) = E_n^{\text{odd}}(\alpha \pm 1), \tag{17}$$

$$E_n^{\text{odd/even}}(\alpha) = E_n^{\text{odd/even}}(\alpha \pm 2). \tag{18}$$

Consequently, it is sufficient to consider diagonalization of the Hamiltonian in any one of the subspaces.

For numerical calculations we need to choose a basis in the state space. We choose it to be the eigenbasis of $H(\alpha = 0)$ so that for small values of α one has a handle over the eigenvalues. Further, the fact that the basis states are independent of α simplifies the numerical computations.

The eigenenergies when $\alpha = 0$ are given by

$$E(m,n) = \frac{\pi^2}{8}\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right] \tag{19}$$

and the eigenstates are (m,n positive), for $\mathcal{H}_{\text{even}}$,

$$\Psi_{m,n}(x,y) = A \sin\left(\frac{m\pi x}{2a}\right) \sin\left(\frac{n\pi y}{2b}\right), \quad m \text{ even}, \quad n \text{ even}, \tag{20}$$

$$\Psi_{m,n}(x,y) = A \cos\left(\frac{m\pi x}{2a}\right) \cos\left(\frac{n\pi y}{2b}\right), \quad m \text{ odd}, \quad n \text{ odd}, \tag{21}$$

and for $\mathcal{H}_{\text{odd}}$,

$$\Psi_{m,n}(x,y) = A \sin\left(\frac{m\pi x}{2a}\right) \cos\left(\frac{n\pi y}{2b}\right), \quad m \text{ even}, \quad n \text{ odd}, \tag{22}$$

$$\Psi_{m,n}(x,y) = A \cos\left(\frac{m\pi x}{2a}\right) \sin\left(\frac{n\pi y}{2b}\right), \quad m \text{ odd}, \quad n \text{ even}, \tag{23}$$

where $A(= 1/\sqrt{ab})$ is the normalization factor. For $\alpha \neq 0$, we diagonalize the full Hamiltonian [Eq. (6)] in a basis spanned by the eigenstates of the rectangular billiard. Because of the observations above, we need to consider only the even or the odd subspaces. We choose to diagonalize $H(\alpha)$ in the odd subspace for the following reason.

The singular nature of the potential when $\alpha \neq 0$ implies that the exact wave functions must vanish at the origin. In fact, they must vanish as $r^{|\alpha|}$ as $r \to 0$. Only in the odd subspace do the basis states vanish as $r \to 0$, in fact, faster than required for $\mid \alpha \mid\ \leq\ 1$. We can, therefore, expect faster improvement in the eigenenergies as basis size is increased. Since the basis states chosen are eigenstates of $H(0)$, we limit the diagonalization to values of α ranging from -1 to $+1$.

To summarize, we diagonalize H(α) in the basis in the odd subspace for α between -1 and $+1$. The results for α between -1 and 0 correspond to the diagonalization of $H(\alpha + 1)$ in the even subspace.

For the numerical calculations we have chosen $a = 1, b = \pi/3$. The results have been checked for the other values of a and b. For a given size of the basis we find that typically about 30–40 % of the eigenvalues show decent convergence with respect to changes of basis size. We have taken basis sizes of 200, 338, 450, and 800. Comparison of 450 and 800 shows that first 350 eigenvalues change only within 1% or less. So we select 350 eigenvalues determined from the larger set and do the NNS analysis. The spectral flow as a function of α ($-1 \leq \alpha \leq 1$) is shown in Fig. 1 for the lowest eigenvalues. It is obvious that the levels are correlated as evidenced by the occurrence of avoided crossings. The spectrum has a reflection symmetry around $\alpha = 0$ when the range of α is chosen to be $-1 \leq \alpha \leq 1$ as expected.

The results of the NNS analysis are shown in Fig. 2 in the range $0 \leq \alpha \leq 1$ for some selected values of α. The following interesting features can be easily seen: (i) At $\alpha = 0$ the $P(s)$ distribution fits the Poisson distribution as expected since the system is integrable. (ii) Again at $\alpha = 1$, the $P(s)$ distribution closely follows the Poisson distribution though the fit is not as good as at $\alpha = 0$ due to the numerical accuracy of the eigenvalues. This behavior is expected from Eq. (17) and Eq. (18) since the spectrum of states at $\alpha = 1$ in the odd basis is the same as the spectrum of states in the even basis but at $\alpha = 0$, as though the flux line is not present. This is, of course, also true in general for all integer values of α. (iii) For generic values of α the $P(s)$ distribution lies in between Poisson and Wigner distributions. At $\alpha = 1/2$ the distribution is

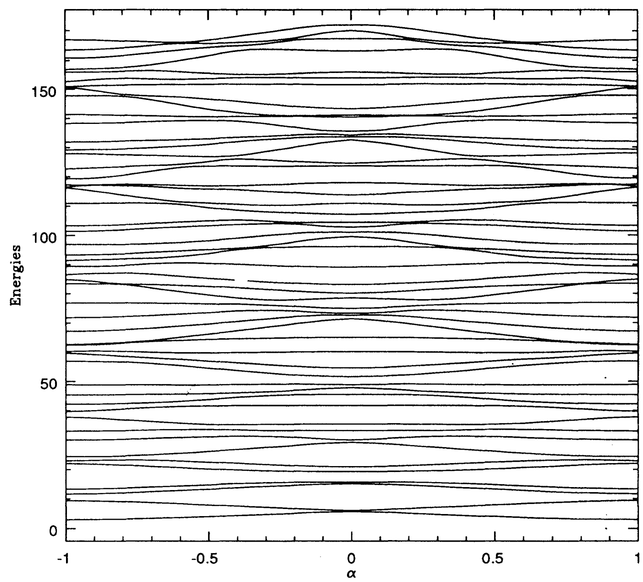

FIG. 1. The spectrum of odd states as a function of α. First 50 energy eigenvalues are shown. The spectrum of states in the even basis may be obtained from this by shifting α to $1 + \alpha$.

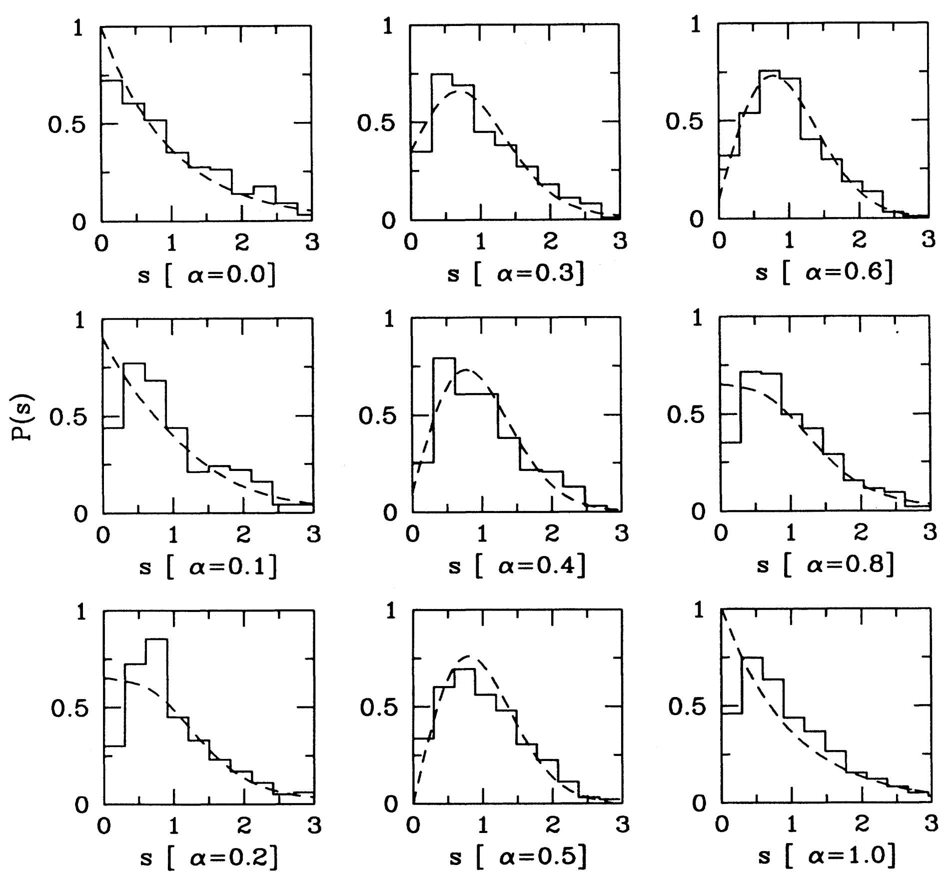

FIG. 2. The NNS distribution for the spectrum of odd states for various values of α as indicated against each figure. The continuous line corresponds to the fit obtained from a linear combination of Poisson and Wigner distributions.

closest to Wigner distribution. This fact coupled with the interpolation property outlined above for integer values of α, immediately suggests a form of $P(s)$ distribution, namely,

$$P(s) = a\ N_p\ e^{-\frac{s}{\sigma_p}}\ +\ (1-a)\ N_w\ s\ e^{-\frac{s^2}{\sigma_w}}, \qquad (24)$$

where $N_i,\ \ i = p, w$ is the normalization factor for Poisson and Wigner distributions and a is obtained by fitting the $P(s)$ to the actual histogram plots in Fig. 2. From the interpolating properties outlined above it is clear that $a = 1$ for integer values of α and most likely $a = 0$ at $\alpha = 1/2$. The form of Eq. (24) suggests that we can always write $a = \cos^2(\eta)$, where $\eta = \eta(\alpha)$. The periodicity in α requires that $\eta(\alpha + 2) = \eta(\alpha) + 2\pi k$, where k is an integer. We make the simplest conjecture that $\eta(\alpha) = \pi\alpha$ with no particular justification. Figure 3 shows that this is a fairly reasonable choice. Of course, at best it is the simplest conjecture consistent with the interpolation properties outlined above, while the allowed parameter space admits other choices. Therefore, the $P(s)$ distribution for the rectangular billiard in the presence of the flux line may be analytically written as

$$P(s) = \cos^2(\pi\alpha)\ N_p\ e^{-\frac{s}{\sigma_s}}\ +\ \sin^2(\pi\alpha)\ N_w\ s\ e^{-\frac{s^2}{\sigma_w}}. \qquad (25)$$

This fit works best for $\alpha \leq 0.5$ where the eigenvalues are also most accurate. It is, therefore, conceivable that this conjecture works for all α. While we did try several other forms like $s^\gamma e^{-s^\beta/\sigma}$, the above form gave good fits consistent with the constraints imposed by the interpolation properties.

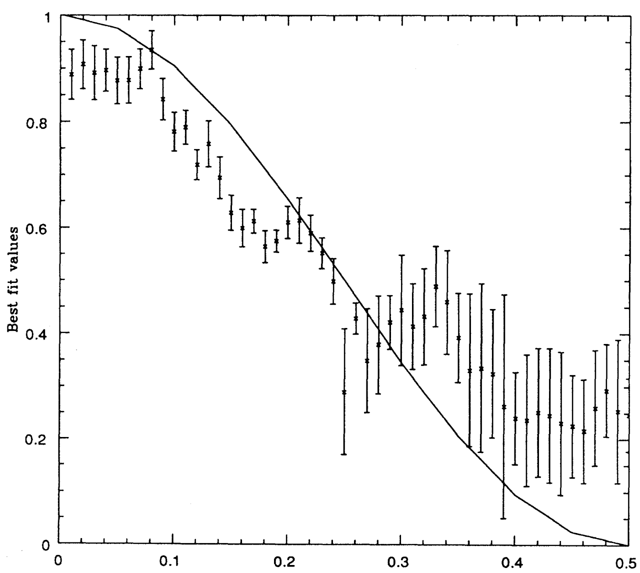

FIG. 3. The value of the fit parameter a, when the histograms of Fig. 2 are fitted to a normalized linear combination of Poisson and Wigner distributions. This is compared with $\cos^2(\pi\alpha)$ shown by the continuous curve.

It is interesting to note that the maximum deviation from the generic distribution for the rectangular billiard occurs at $\alpha = 1/2$ because of the $\sin^2(\pi\alpha)$ factor. It is curious to note that it is the same factor that appears in the Aharanov-Bohm scattering where the scattering cross section for a plane wave by a singular magnetic flux line is proportional to $\sin^2(\pi\alpha)/\cos^2(\theta/2)$, where θ is the scattering angle. Again the effect of scattering is maximum at $\alpha = 1/2$. It is however not clear what the connection is. It may be guessed that because of the interpolation property in α, the α dependence of the properties of the system must be through a periodic function.

Another point to note is that the large s behavior of the NNS distributions is dominated by the tail of the Poisson distribution when α is away from the value $\alpha = 1/2$. Finally, all the above statements apply to the spectrum of even states due to the symmetry of the system. The $P(s)$ in the odd basis at some value of α goes over to $P(s)$ in the even basis corresponding to $1 + \alpha$.

IV. DISCUSSION

To summarize, we have analyzed a Hamiltonian system, which is well known in several other contexts, which describes the dynamics of a rectangular billiard in the presence of a flux line. Even though the classical equations of motion remain unchanged, the configuration space of the system being a rectangle with its origin removed makes the system pseudointegrable. Classically this implies that the dynamics remains unchanged except for the set (of measure zero) of orbits passing through the origin. This is brought about more succinctly in the classical phase space analysis. The particle plus flux composite system, however, shows nontrivial behavior through the quantum spectrum and is a function of the coupling of the flux line to the particle.

While it is correct to say that the spectrum depends nontrivially on the coupling α, the changes in the NNS distribution depend crucially on the type of boundary chosen. We have, therefore, also studied the NNS distribution under different boundary conditions, like the boundary removed with an oscillator confinement and particle confined to a circular disk instead of a rectangle. All these systems may be regarded as pseudointegrable in the same sense as the particle confined to a rectangle. However, the NNS distributions are dramatically different in each of these cases:

(i) In the case of the Hamiltonian with boundary removed and an oscillator term added, the system is still pseudointegrable classically and the quantum spectrum is known exactly. The NNS carried out in each partial wave, however, does not deviate from the case when $\alpha = 0$. In fact as is well known it is neither a Poisson distribution nor a Wigner distribution. In a way this is obvious since the spectrum can be obtained from the large distance behavior which washes out the small distance behavior where pseudointegrability shows up.

(ii) If we take a circular disk instead of the rectangular boundary the spectrum is obtained as the zeros of the Bessel function $J_{|l-\alpha|}$ in each partial wave. The spectrum, therefore, has nontrivial behavior as a function of α. However, the NNS distribution is essentially the same as when $\alpha = 0$, that is, it is a δ function as in the case of a one-dimensional oscillator.

Note that in both of the above cases NNS statistics is done in each partial wave, which effectively reduces the system to a one-dimensional system. We have chosen to analyze the NNS in each symmetry class (or each partial wave) separately as in the case of the rectangular billiard so that we are able to compare the two systems. In general, however, a Poisson distribution may be obtained for such systems only when the spectrum includes all the symmetry classes. There is no effect on the NNS statistics in the above examples even though the spectrum is α dependent. The effect of the flux line on the spectrum however shows differently in these systems. It has been shown recently [13] that the presence of the flux line introduces nontrivial changes in the density states of the system. In particular the oscillating part of the density of states of the system cannot be completely explained using the simple periodic orbit theory (POT). One will have to modify the standard POT to include quantum $\hbar$ corrections in order to explain the density of states. Interestingly enough the amplitude of the oscillating part of the density states is again related to the cross section of Aharanov-Bohm scattering.

(iii) The case of Aharanov-Bohm billiard in complicated enclosures has been analyzed before in [10]. While the system continues to be pseudointegrable, the NNS distribution is either the one corresponding to GOE or the Gaussian unitary ensemble (GUE). Here it appears that the boundary effects on the NNS distribution are much stronger than the pseudointegrability.

(iv) Our system with rectangular boundary is a hybrid. Unlike the above examples it is neither reducible to an effectively one-dimensional system nor are the boundary effects too strong. The NNS distribution thus reflects the pseudointegrability aspect primarily. As seen in the last section, however, it does show nongeneric features.

It appears, therefore, that the cases where NNS shows a relation to pseudointegrability are those for which the system is neither reducible effectively to a one-dimensional system nor are the boundary effects too strong.

Therefore, the connection between classical properties such as integrability, chaos, and their quantum manifestations remains elusive. While in the case of classically chaotic systems, with or without T invariance, there appears to be a corresponding universal quantum signature in terms of their universal $P(s)$ distributions (GOE or GUE) the same type of connection does not seem to exist for pseudointegrable and probably integrable systems.

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